Generalized Descents and Normality

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Abstract

We use Janson's dependency criterion to prove that the distribution of d-descents of permutations of length n converge to a normal distribution as n goes to infinity. We show that this remains true even if d is allowed to grow with n, up to a certain degree.

1 Introduction

Let $p = p_1 p_2 \cdots p_n$ be a permutation. We say that the pair (i, j) is a d-descent in p if $i < j \leq i+d$, and $p_i > p_j$. In particular, 1-descents correspond to descents in the traditional sense, and $(n-1)$-descents correspond to inversions. This concept was introduced in [2] by De Mari and Shayman, whose motivation came from algebraic geometry. They have proved that if n and d are fixed, and c_k denotes the number of permutations of length n with exactly k d-descents, then the sequence $c_0, c_1, \cdots$ is unimodal, that is, it increases steadily, then it decreases steadily. It is not known in general if the sequence $c_0, c_1, \cdots$ is log-concave or not, that is, whether $c_{k-1}c_{k+1} \leq c_k^2$ holds for all k. We point out that in general, the polynomial $\sum_k c_k x^k$ does not have real roots only. Indeed, in the special case of $d = n-1$, we get the well-known [1] identity

$$\sum_k c_k x^k = (1+x) \cdot (1+x+x^2) \cdot \cdots \cdot (1+x+\cdots+x^{n-1}),$$

which has all nth roots of unity as roots. Indeed, in this case, a d-descent is just an inversion, as we said above.

In this paper, we prove a related property of generalized descents by showing that their distribution converges to a normal distribution as the length n of our permutations goes to infinity. Our main tool is *Janson's dependency criterion*, which is a tool to prove normality for sums of bounded random variables with a sparse dependency graph.

2 The Proof of Asymptotic Normality

2.1 Background and Definitions

We need to introduce some notation for transforms of the random variable Z. Let $\bar{Z} = Z - E(Z)$, let $\tilde{Z} = \bar{Z}/\sqrt{\mathrm{Var}(Z)}$, and let $Z_n \to N(0,1)$ mean that Z_n converges in distribution to the standard normal variable.

For the rest of this paper, let $d \geq 1$ be a fixed positive integer. Let $X_n = X_n^{(d)}$ denote the random variable counting the d-descents of a randomly selected permutation of length n. We want to prove that X_n converges to a normal distribution as n goes to infinity, in other words, that $\tilde{X}_n \to N(0,1)$ as $n \to \infty$. Our main tool in doing so is a theorem called Janson's dependency criterion. In order to state that theorem, we need the following definition.

Definition 1 *Let $\{Y_{n,k}|k = 1, 2\cdots\}$ be an array of random variables. We say that a graph G is a* dependency graph *for $\{Y_{n,k}|k = 1, 2\cdots\}$ if the following two conditions are satisfied:*

1. *There exists a bijection between the random variables $Y_{n,k}$ and the vertices of G, and*
2. *If V_1 and V_2 are two disjoint sets of vertices of G so that no edge of G has one endpoint in V_1 and another one in V_2, then the corresponding sets of random variables are independent.*

Note that the dependency graph of a family of variables is not unique. Indeed if G is a dependency graph for a family and G is not a complete graph, then we can get other dependency graphs for the family by simply adding new edges to G.

Now we are in position to state Janson's dependency criterion.

Theorem 1 *[5] Let $Y_{n,k}$ be an array of random variables such that for all n, and for all $k = 1, 2, \cdots, N_n$, the inequality $|Y_{n,k}| \leq A_n$ holds for some*

real number A_n, and that the maximum degree of a dependency graph of $\{Y_{n,k}|k = 1, 2, \cdots, N_n\}$ is Δ_n.

Set $Y_n = \sum_{k=1}^{N_n} Y_{n,k}$ and $\sigma_n^2 = Var(Y_n)$. If there is a natural number m so that

$$N_n \Delta_n^{m-1} \left(\frac{A_n}{\sigma_n}\right)^m \to 0, \tag{1}$$

then

$$\tilde{Y}_n \to N(0, 1).$$

2.2 Applying Janson's Criterion

We will apply Janson's theorem with the $Y_{n,k}$ being the indicator random variables $X_{n,k}$ of the event that a given ordered pair of indices (indexed by k in some way) form a d-descent in the randomly selected permutation $p = p_1 p_2 \cdots p_n$. So N_n is the number of pairs (i, j) of indices so that $1 \leq i < j \leq i + d \leq n$. Then by definition,

$$Y_n = \sum_{k=1}^{N_n} Y_{n,k} = \sum_{k=1}^{N_n} X_{n,k} = X_n.$$

There remains the task of verifying that the variables $Y_{n,k}$ satisfy all conditions of Jansen's theorem.

First, it is clear that $N_n \leq nd$, and we will compute the exact value of N_n later. By the definition of indicator random variables, we have $|Y_{n,k}| \leq 1$, so we can set $A_n = 1$ for all n.

Next we consider the numbers Δ_n in the following dependency graph of the family of the $Y_{n,k}$. Clearly, the indicator random variables that belong to two pairs (i, j) and (r, s) of indices are independent if and only if the sets $\{i, j\}$ and $\{r, s\}$ are disjoint. So fixing (i, j), we need one of $i = r$, $i = s$, $j = r$ or $j = s$ to be true for the two distinct variables to be dependent. So let the vertices of G be the N_n pairs of indices (i, j) so that $i < j \leq i + d$, and connect (i, j) to (r, s) if one of $i = r$, $i = s$, $j = r$ or $j = s$ holds. The graph defined in this way is clearly a dependency graph for the family of the $Y_{n,k}$. For a fixed pair (i, j), each of these four equalities occurs at most d times. (For instance, if $i = s$, then r has to be one of $i - 1, i - 2, \cdots, i - d$.) Therefore, $\Delta_n \leq 4d$.

If we take a new look at (1), we see that the Janson criterion will be satisfied if we can show that σ_n is large. This is the content of the next lemma.

Lemma 1 *If $n \geq 2d$, then*

$$Var(X_n) = \frac{6dn + 10d^3 - 3d^2 - d}{72}. \tag{2}$$

In particular, $Var(X_n)$ is a linear *function of n.*

Note that in particular, for $d = 1$, we get the well-known fact [1] that the variance of Eulerian numbers in permutations of length n is $(n+1)/12$.
Proof: By linearity of expectation, we have

$$\begin{aligned}
\mathrm{Var}(X_n) &= E(X_n^2) - (E(X_n))^2 &(3)\\
&= E\left(\left(\sum_{k=1}^{N_n} X_{n,k}\right)^2\right) - \left(E\left(\sum_{k=1}^{N_n} X_{n,k}\right)\right)^2 &(4)\\
&= E\left(\left(\sum_{k=1}^{N_n} X_{n,k}\right)^2\right) - \left(\sum_{k=1}^{N_n} E(X_{n,k})\right)^2 &(5)\\
&= \sum_{k_1,k_2} E(X_{n,k_1}X_{n,k_2}) - \sum_{k_1,k_2} E(X_{n,k_1})E(X_{n,k_2}) &(6)
\end{aligned}$$

Clearly, $E(X_{n,k}) = 1/2$, so the N_n^2 summands that appear in the last line of the above chain of equations with a *negative sign* are each equal to $1/4$. As far as the N_n^2 summands that appear with a positive sign, *most* of them are equal to $1/4$. More precisely, if X_{n,k_1} and X_{n,k_2} are independent, then

$$E(X_{n,k_1}X_{n,k_2}) = E(X_{n,k_1})E(X_{n,k_2}) = \frac{1}{4}.$$

If $k_1 = k_2$, then $E(X_{n,k_1}X_{n,k_2}) = E(X_{k_1}^2 = E(X_{k_1}) = 1/2$. Otherwise, if X_{n,k_1} and X_{n,k_2} are dependent, then either $E(X_{n,k_1}X_{n,k_2}) = 1/3$, or $E(X_{n,k_1}X_{n,k_2}) = 1/6$. Indeed, if X_{k_1} is the indicator variable of the pair (i,j) being a d-descent and X_{k_2} is the indicator variable of the pair (r,s) being a d-descent, then as we said above, X_{n,k_1} and X_{n,k_2} are dependent if and only if one of $i = r$, $i = s$, $j = r$ or $j = s$ holds. If $i = r$ or $j = s$ holds, then $E(X_{n,k_1}X_{n,k_2}) = 1/3$, and if $i = s$ or $j = r$ holds, then $E(X_{n,k_1}X_{n,k_2}) = 1/6$. Indeed, for instance, with $i = r$, we have $X_{n,k_1} = X_{n,k_2} = 1$ if and only if p_i is the largest of the entries p_i, p_j, and p_s. Similarly, with $i = s$, we have $X_{n,k_1} = X_{n,k_2} = 1$ if and only if $p_r > p_i > p_j$.

We will now count how many summands $E(X_{n,k_1}X_{n,k_2})$ are equal to $1/2$, to $1/3$, and to $1/6$.

1. First, $E(X_{n,k_1}X_{n,k_2}) = 1/2$ if and only if $k_1 = k_2$. This happens N_n times, once for each pair (i,j) so that $i < j \leq i+d$. For a given i, there are d such pairs if $i \leq n-d$, and $d-t$ such pairs if $i = n-d+t$, so

$$N_n = (n-d)d + (d-1) + (d-2) + \cdots + 1 = (n-d)d + \binom{d}{2}.$$

2. Second, $E(X_{n,k_1}X_{n,k_2}) = 1/3$ if $i = r$, or $j = s$. By symmetry, we can consider the first case, then multiply by two. If $i \leq n-d$, then we have $d(d-1)$ choices for j and s, and if $i = n-d+t$, then we have $(d-t)(d-t-1)$ choices. So the number of pairs (k_1,k_2) so that $E(X_{n,k_1}X_{n,k_2}) = 1/3$ is

$$2(n-d)d(d-1) + 2(d-1)(d-2) + 2(d-2)(d-3) + \cdots + 2\cdot 2\cdot 1 =$$

$$2(n-d)d(d-1) + 4\binom{d}{3}.$$

3. Finally, $E(X_{n,k_1}X_{n,k_2}) = 1/6$ if $i = s$, or $j = r$. By symmetry, we can again consider the first case, then multiply by two. If $d \leq i \leq n-d$, then there are d^2 choices for (j,r). If $i \leq d$, then there are d choices for j, and $i-1$ choices for r. If $n-d < i$, then there are $n-i$ choices for j, and d choices for r, assuming that $n \geq 2d$. So the number of pairs (k_1,k_2) so that $E(X_{n,k_1}X_{n,k_2}) = 1/6$ is

$$2(n-2d)d^2 + 2(d-1)d + 2(d-2)d + \cdots + 2d = 2(n-2d)d^2 + d^2(d-1).$$

   For all remaining pairs (k_1,k_2), the variables X_{n,k_1} and X_{n,k_2} are independent, and so $E(X_{n,k_1}X_{n,k_2}) = 1/4$.

Comparing our results from cases 1-3 above with (3), and recalling that in all other cases, $E(X_{n,k_1}X_{n,k_2}) = 1/4$, we obtain the formula that was to be proved. $\diamond$

The proof of our main theorem is now immediate.

Theorem 2 *Let d be a fixed positive integer. Let X_n be the random variable counting d-descents of a randomly selected n-permutation. Then $\tilde{X}_n \rightarrow N(0,1)$.*

Proof: Use Theorem 1 with $Y_n = X_n$, $\Delta_n = 4d$, $N_n = (n-d)d + \binom{d}{2}$, and $\sigma_n = \sqrt{\frac{6dn+10d^3-3d^2-d}{72}}$. All we need to show is that there exists a positive integer m so that

$$\left((n-d)d + \binom{d}{2}\right) \cdot (4d)^{m-1} \cdot \left(\frac{72}{6dn+10d^3-3d^2-d}\right)^{m/2} \to 0,$$

for which it suffices to find a positive integer m so that

$$(dn) \cdot (4d)^{m-1} \cdot \left(\frac{12}{dn}\right)^{m/2} \to 0. \tag{7}$$

Clearly, any $m \geq 3$ suffices, since for any such m, the left-hand side is of the form C/n^{α}, for positive constants C and α. $\diamond$

3 Further Directions

We see from (7) that the statement of Theorem 2 can be strengthened, from a constant d to a d that is a function of n. Indeed, (7) is equivalent to saying that

$$cn \left(\frac{d}{n}\right)^{m/2} \to 0.$$

This convergence holds as long as $d \leq n^{1-\epsilon}$ for some fixed positive ϵ, we can choose m so that $(m/2) \cdot \epsilon > 1$, and then condition (7) will be satisfied. So we have proved the following theorem.

Theorem 3 *Let $n \to \infty$, and let us assume that there exists a positive constant ϵ so that for n sufficiently large, $d = d(n) \leq n^{1-\epsilon}$. Let X_n be defined as before. Then*

$$\tilde{X}_n \to N(0,1).$$

This leaves the cases of larger d open. We point out that in the special case of $d = n-1$, that is, inversions, asymptotic normality is known [3], [4].

Another possible direction for generalizations is the following. Let $\mathbf{d} = (d_1, d_2 \cdots, d_{n-1})$, where the d_i are positive integers. If $p = p_1 \dots p_n$ is in an n-permutation, let $f_d(p)$ be the number of pairs (i,j) such that $0 < j-i \leq d_i$ and $p_i > p_j$. For instance, if $\mathbf{d} = (1,1,\dots,1)$ then $f_d(p)$ is the number of

descents of p. If $\mathbf{d} = (n-1, n-2, ..., 1)$ then $f_d(p)$ is the number of inversions of p. It is known [2], by an argument from algebraic geometry, that if

$$c_k = |\{p \in S_n : f_d(p) = k\}|,$$

then the sequence $c_0, c_1, \cdots$ is unimodal. Log-concavity and normality are not known. Note that in this paper, we have treated the special case of $\mathbf{d} = (d, d, \cdots, d)$.

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